COGNITIVE DIFFICULTIES IN EARLY ALGEBRA

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Over seven months, one group of Year 7 students showed little, if any, progress with algebra while another group improved markedly. The latter were significantly better in developing substitution skills, in identifying when two algebraic expressions were equal, and in understanding algebraic symbols as generalized numbers and/or variables and as representing numbers rather than objects. The former were significantly more inclined to persist with incorrect pre-algebra ideas and to interpret conjoining (e.g., in '2n') as addition. They were not coping with the new ideas being presented to them.

THE KEY TO PROGRESS

An important question, especially from a teacher's point of view, is the question of why some students do not show much or any progress despite the fact that they are present for the same lessons as other students who do progress, some of them quite rapidly. What is it that students really need to learn before they can show signs of general progress? Do they need to reach certain steps in a hierarchy of concepts and/or skills before they can make secure progress in other aspects of learning? One of the ways in which concern about the "slow learners" was addressed was to study their performance sequentially from one test to another in comparison with the "fast learners".

This paper is based on the results of four testing stages for 208 Year 7 students who completed the same test instrument three times during their first three weeks of algebra in 1990, and for the 186 of these who completed it a fourth time some six months after the teaching intervention sessions.

Analyses based on standards. To set criterion levels for following changes in performance from one test to the next, four equal ranges up to and including the maximum scored by these Year 7 students were chosen: 0 to 13, 14 to 27, 28 to 41 and 42 to 55. Scores within these ranges were called Standards One to Four and will be referred to as S1, S2, S3 and S4 respectively. The frequency distribution of students in each of these standard brackets is displayed in Table 1.

Table 1: Frequency Distribution of Year 7 Students in Standards Brackets

Standard	Range of Scores	Test 1	Test 2	Test 3	Test 4
S1	0 - 13	158	71	24	23
S2	14 - 27	42	103	111	67
S3	28 - 41	8	30	59	76
S4	42 - 55	0	4	14	20
Totals		208	208	208	186

An investigation was carried out to find out if any students stayed at the lowest level (S1) in all four tests, and it was found that 15 actually did. There were two more students who were absent for the fourth test but remained in the S1 group for the first three tests. These 17 students, then, had shown very little, if any, improvement during the time period of the testing program, even though it included at least three weeks of introductory algebra teaching. Another 7 students stayed in the S2 bracket for all four tests. There were 18 students who, in contrast, had moved from either the S1 bracket (in 8 cases) or the S2 bracket (in 10 cases) to the top S4 bracket during the same period. The question that called for an answer was whether or not some vital aspect of learning allowed the latter two groups of students to progress so markedly while the former two groups made little headway.

DEVELOPMENTAL CONTRASTS BETWEEN TWO EXTREME GROUPS

An investigation of the contrasts in learning patterns of the two extreme groups shed light on the problem of why some improve and others do not. The groups were the 17 students who stayed in the S1 bracket (referred to as the SS11 Group) and the 8 who moved from the S1 bracket to the top S4 bracket (the SS14 Group). Table 2 presents a summary of the relevant statistics for Tests 1 and 2.

Table 2: Summary of *t*-tests for Groups SS11 and SS14 on Tests 1 & 2 Responses

Test	Scale	Comment	Max	Mean SS11	Mean SS14	*t* value	*df* V	*p*	Favours
1	Total	Test Total	65	5.12	9.25	3.86	23	***	SS14
2	GNV	Gen.No.and/ or Variable	17	1.33	5.83	5.49	5.70	**	SS14
2	Total	Test Total	65	8.88	20.50	4.51	8.08	**	SS14
2	SUBS	Substitute & Solve	7	0.38	5.00	4.41	7.22	**	SS14
2	EQL	Equals Scale	9	0.11	1.67	3.57	5.70	*	SS14
2	VBL	Variable	11	0.10	2.17	3.39	5.28	*	SS14
2	NBR	Number View	4	0.43	1.67	2.49	18	*	SS14
2	AD	"*a, d*" Scale	2	0.18	1.00	2.34	17	*	SS14
2	CON	Conjoining	7	2.31	0.43	-3.38	21.00	**	SS14
2	PRE	Prestructural View	19	15.50	8.67	-3.10	10	*	SS14

Note: Test 2 entries sorted in order of *t* values. Max. = maximum possible score (= no. of items for scales). V *df* : decimal point if using separate variances; otherwise, pooled variance. *** $p \leq .001$, ** $.001 < p \leq .010$, * $.010 < p \leq .050$.

Table 2 opened the door for a view of what aspects of learning made the difference between these two groups after their first week and a half of algebra. In Test 1, the SS14 Group scored significantly better on test total than the SS11 Group, although all members of both groups were at Standard 1 (test scores less than 14), and the difference between the mean scores was only about 4 points. There were no other significant differences at this pre-algebra stage. The *t*-tests based on Test 2 responses identified that those who were destined to progress to the top Standard 4 level significantly out-scored those who stayed in the bottom Standard 1 level in the following ways:

1. Their average test score was more than double that attained by Group SS11 and was about 12 points better (out of 65);

2. They had progressed significantly further in their understanding of algebraic symbols as generalized numbers and/or variables (GNV Scale), e.g., symbols allowed at least two values in equations '$2x + y = 9$' and/or '$c + d = 10$', and when comparing '$t + t$' with '$t + 4$';

3. They were more rapidly developing the skills for substituting a numerical value into algebraic expressions and for solving a simple equation (SUBS Scale), e.g., If '$y = 3$, $2y + 5 = ...$, $2(y + 5) = ...$, $2(5y) = ...$ ', and 'Solve $3a= 36$';

4. They were more successful in identifying when two algebraic expressions were equal (EQL Scale), e.g., '$a + b + c = a + x + c$', '$2a + 3b + 7 = 5a + 7$', '$a + 2b + 2c = a + 2b + 4c$';

5. They were developing the variable concept more rapidly (VBL Scale), e.g., comparing values of '$2n$' and '$n + 2$' and/or '$t + t$' and '$t + 4$';

6. They had the greater tendency to regard the letter symbols in early algebra as standing for numbers rather than objects (NBR Scale), e.g., for the Professors-and-Students problem, interpreting 'S' as "number of students" in the equation '$S = 6P$';

7. They were more able to allow the symbols 'a' and 'd' to stand for numbers without restriction (AD Scale) in the question "If 'a' and 'd' are any two numbers, which, if either, is the bigger? Give a reason for your answer.";

8. They were less inclined to conjoin symbols for addition (CON Scale), e.g., If $y = 3$, $2y = 6$, not 5, and "Add 4 onto $3n$" gives '$4 + 3n$ ', not '$7n$'; and

9. They were breaking away more rapidly from some of their prestructural views of algebra (PRE Scale), e.g., claiming that '$t + t$' is always greater than '$t + 4$', ordering symbols alphabetically, and/or needing numerical values for all symbols.

Findings 2, 4, 5, 6, and 7 all, in some degree, recorded that those in the group on the verge of greater improvement had started to develop the concept that the alphabetic symbols of early algebra represented numbers and that the numbers could vary. Those not destined to improve out of Standard 1 over the time period used for the collection of research data lagged significantly behind them and had failed to appreciate that the symbols that had been introduced to them were standing for numbers which could vary. This seems to be empirical evidence that it was the student views of the meaning of the symbols which identified significant differences between those who were on the way to improvement and those who were not. This outcome made logical sense considering the fact that most of the test items were designed to measure the level of understanding of symbols. Findings 2, 4 and 5 indicated that those who were to progress the more had started to understand that algebraic symbols represented numerical variables, and they had begun to apply this concept to some, at least, of the problems presented in the test.

Finding 3 above recorded that those on the way to higher scores were faster in developing the skills needed for solving a simple equation ('$3a = 36$'), and for substituting a given value for 'y' in given expressions containing y (such as '$2y + 5$'), showing that they had a clearer grasp of the meanings of algebraic expressions. Findings 8 and 9 above reported that it was helpful for the prospect of improving in algebra to understand the convention that conjoining was used for multiplication in algebra, and to start to understand what the test questions were actually asking. Those who were not on the way to progress made very little intelligent headway at the time of Test 2 on most of the items in the Prestructural Scale, as shown by their high mean score (15.50 out of a possible 19) on the PRE Scale, which tallied the number of times the meaning of a problem was missed.

Data obtained in Test 3 allowed the continuation of the investigation of what aspects of learning discriminated between those who proceeded to Standard 4 from those who stayed at Standard 1. The differences between the means of these two groups were significant on 22 scales, the large number emphasizing the growing gap between the rates of development within the two groups. In every case the differences favoured the SS14 Group. At the time of responding to Test 4, the SS14 Group were still well ahead of the SS11 Group, as there were 21 significant differences between the means.

Comparison of Groups SS11 and SS14 Over Four Tests. Figures 1 and 2 summarize the scores per item for Groups SS11 and SS14 respectively on scales which registered significant differences, using *t*-tests, for Tests 2, 3, and 4.

Figure 1 records that Group SS11 showed very little change in their views about algebra. The average scores in the four tests for each scale are quite close together, indicating that there was not much change from test to test, except for an improvement on the Substitute and Solve (SUBS) Scale and the 'a, d' (AD)Scale. They also recorded just minimal changes in their incorrect views, as measured by the Conjoin (CON) Scale and the Prestructural (PRE) Scale.

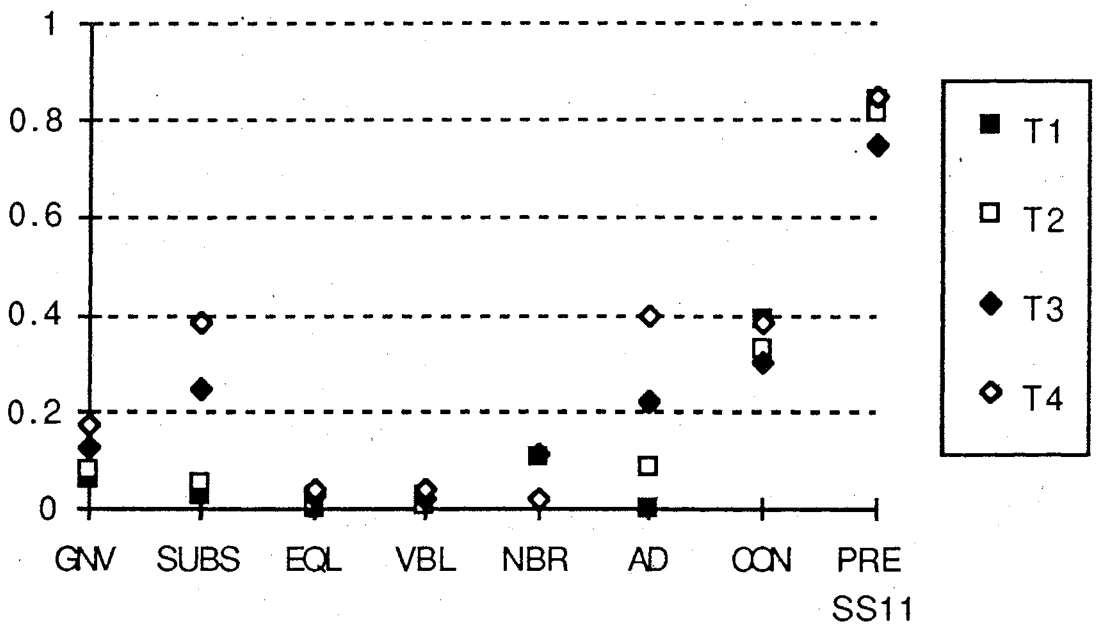

Figure 1: Average scores/item for SS11 Group

Figure 2 reports that the SS14 Group, on the other hand, showed a rapid improvement on the first six scales, especially the Generalized Number and/or Variable (GNV) Scale, the Substitute and Solve (SUBS) Scale and the '*a*, *d*' (AD) Scale. They showed a growth in their ability to solve equality problems (EQL Scale), and in understanding algebraic symbols as variables (VBL Scale), and as representing numbers (NBR Scale). They moved away from incorrect views, as recorded by the noticeable decrease in average scores per item on the Conjoin (CON) Scale and the Prestructural (PRE) Scale.

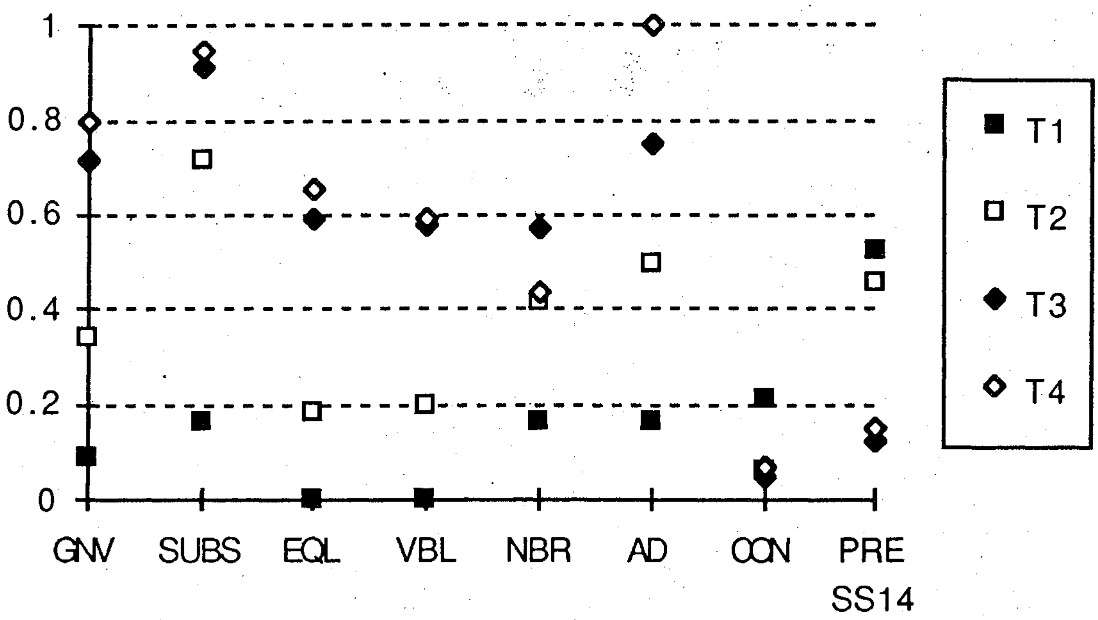

Figure 2: Average scores/item for SS14 Group

The data used in the search for the key to progress were found to be applicable to two of the propositions which were supported by analyses of the frequencies of successful responses from all 517 students in the study. These are discussed in turn.

Proposition 1. Empirical support was given to Proposition 1 which claimed that *success in interpreting algebraic expressions is a prelude to success in comparing the values of two expressions or comparing the values of two variables within the one expression.*

In order to make substitutions successfully, students had to know the conventions for writing the expressions so that they could interpret what the expressions meant. An investigation was carried out to see how the two groups compared on a subset of the SUBS Scale, namely the SUB Substitution Scale which was available as a measure of skill in substitution. As was expected, *t*-tests using scores on the SUB Scale showed that there was no significant difference between the two groups in Test 1 but that the differences were significant in the other tests, just as was the case for the SUBS Scale. A summary of *t*-test analyses for the SUB Scale responses is presented in Table 3.

Table 3: Summary of t-tests for Groups SS11 and SS14 on SUB Substitution Scale Responses

Test	Scale	Max	Mean SS11	Mean SS14	t value	df V	p	Favours
2	SUB	6	0.24	4.38	5.09	8.31	***	SS14
3	SUB	6	1.76	5.75	7.33	18.92	***	SS14
4	SUB	6	2.07	5.75	6.64	15.35	***	SS14

Note: Test 2 entries sorted in order of t values. Max. = maximum possible score (= no. of items for scales). V df : decimal point if using separate variances; otherwise, pooled variance. *** $p \leq .001$.

These analyses clearly showed that the SS14 Group progressed to almost 100% efficiency in substitution skills after just three weeks of algebra, as could be judged by their high average scores (5.75) on Tests 3 and 4. Those in the SS11 group achieved only about one-third of that success rate, reaching an average scale score of about "2" in Tests 3 and 4. If Proposition 1 applied to Group SS11 students, then their failure to succeed in interpreting algebraic expressions sufficiently well to enable them to substitute correctly should have been accompanied by failure to succeed with the test items which formed the VBL Variables Scale, a scale which gave a measure of their degree of success with items requiring them to compare the values of two expressions (in Items 10, 12, and 15 (iii)) or two variables within one expression (in Item 6 (c)).

Figure 3 illustrates the way the differential rates of development for Groups SS11 and SS14 followed the expectations expressed in Proposition 1, by graphing the average scores per scale item on the scales in question. The SS14 graphs seem to indicate that progress on substitution is a prelude to progress with problems involving the notion of a numerical variable, in the form required by the items in the VBL Scale. At the same time, the graphs for the SS11 Group show that improvement on the VBL measure does not necessarily follow once there is some improvement on the SUB Scale. These outcomes support the reasoning that one needs to reduce the memory load required for simply understanding the meaning of an algebraic expression before one can carry out cognitive tasks involving relationships between algebraic expressions, as was required in items belonging to the VBL Scale.

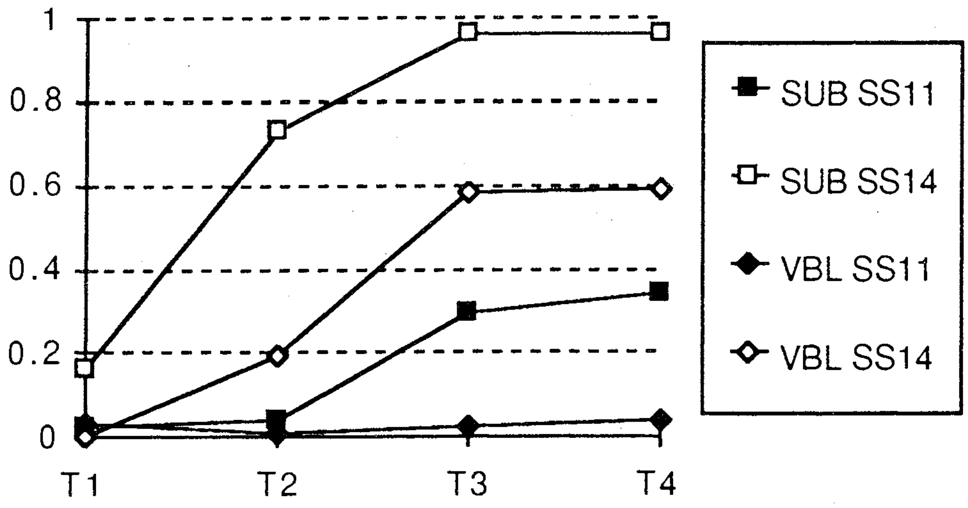

Figure 3: Average scores/item for SS11 and SS14 Groups on SUB and VBL Scales

Proposition 2. Proposition 2 stated that: *Success in solving a simple linear equation is a prelude to success in identifying the conditions for equality of two algebraic expressions or for the equality of two variables within the one equation.*

The EQL Equality Scale found its place in Table 2 and Figures 1 and 2 as one of the measures which identified significant differences in the learning rates of the beginning algebra students in Groups SS11 and SS14 in their first week and a half of algebra. This scale supplied a measure of success in dealing with the equality notion in the contexts mentioned in Proposition 2 and so provided one suitable assessment for testing the proposition with respect to the rates of cognitive development shown by the two groups. One item, namely, Question 8 (b) ("If $3a = 36$, what would be the value of a?"), measured ability to solve a simple algebraic equation and was chosen as the other measure needed for testing Proposition 2. As Question 8(b) was a subset of the EQL Equality Scale, the scale scores were adjusted to form the EQL* Adjusted Equality Scale by subtracting scores on Item 8 (b), so that there would be no overlap between the two measures being considered.

Figure 4 displays the differential rates of development on these two measures for each of the groups SS11 and SS14. Support was evident for Proposition 2. Those who improved their scores on the Equality Scale had, in most cases, previously improved their success with solving a simple equation in Item 8 (b). It was established, however, that success on this item was not a sufficient condition for success on the Equality Scale items. It seemed that the memory load needed for solving the easier cases of algebraic equality had to be reduced before more complex equality problems could be solved. Solving an equation involving one arithmetical operation and one variable was apparently less demanding cognitively than finding the condition for the equality of two algebraic expressions.

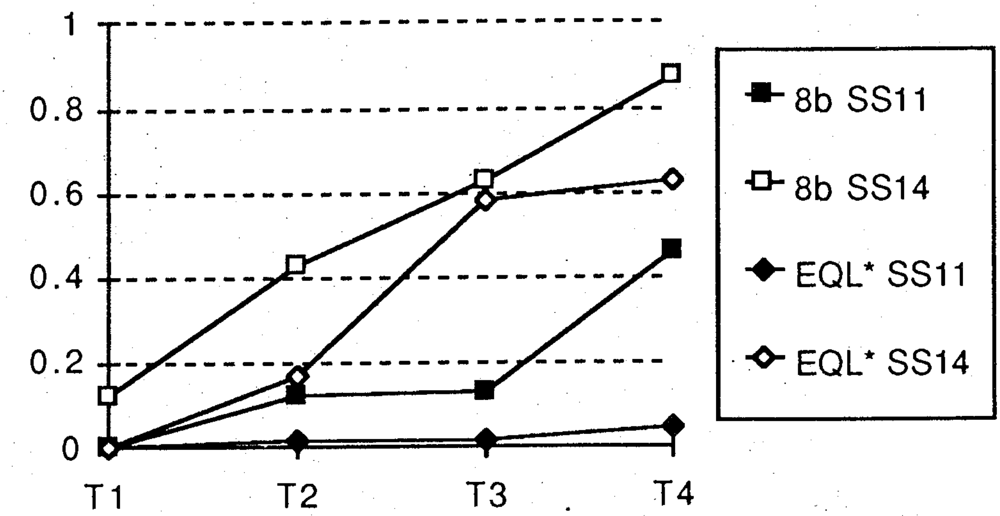

Figure 4: Average scores/item for SS11 and SS14 Groups on EQL* Scale and Q.8b

Persistence of Incorrect Ideas. Two other variables from Table 2 deserve close attention in this exploration of the problem of why some students progress while others do not. They were scores on the CON Conjoining Scale and the PRE Prestructural Scale, both measures of incorrect thinking about early algebra. There were no test items in common between these two scales and, between them, they covered 26 items. The scale averages for the two groups, as graphed in Figure 5, indicated that members of the SS11 Group were more persistent in their incorrect views than were their counterparts in the SS14 Group. This is an additional insight into the factors which influenced the vastly different rates of development recorded for these two groups.

Those in the SS11 Group hardly changed, on average, in their acceptance of the incorrect view of conjoining: Only six of these 17 students kept their CON Scale score under "3" (out of 7) for Tests 3 and 4. They also showed very little change in their prestructural approaches to the 19 problems registered in the PRE Scale. Many had missing data classification for the scale and all the registered scores were "11" or more, except in two cases. It seemed that the class activities were making very little impact on their way of thinking.

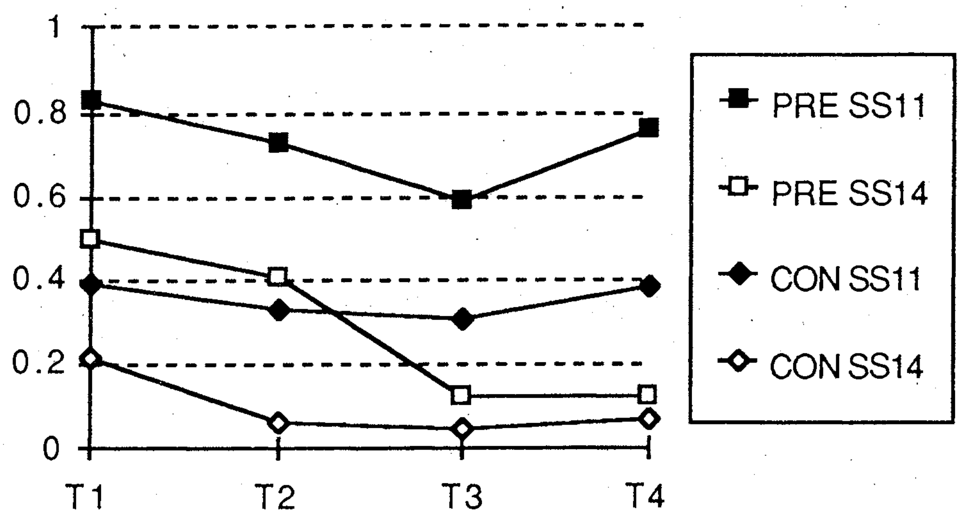

Figure 5: Average Scores/item for SS11 and SS14 Groups on PRE and CON Scales

In contrast, those who made rapid strides in mastering the basic concepts of early algebra, the members of Group SS14, distanced themselves from the misconceptions measured by these two scales. None of them, for instance, scored greater than "1" on the CON Scale after Test 2. Only two scored more than "4" on the PRE Scale after Test 2 and all but one of the rest scored "2", "1" or "0" after Test 2.

Numbers View Versus Objects View. The remaining feature of Table 2 to merit comment is the entry which recorded that students in Group SS14 were significantly more inclined to view algebraic symbols as representing numbers than were the members of Group SS11. The contrast between the two groups in terms of their tendencies to regard the symbols as standing for numbers or as standing for objects (or people) is brought out by the graphs in Figure 6. By Tests 3 and 4 the difference between the groups on the OBJ Objects View Scale was statistically significant and the difference on the NBR Numbers View Scale continued to be statistically significant.

Figure 6 displays the facts that the SS11 Group scarcely changed their point of view at all about what algebraic symbols basically represented: they persistently kept, on average, to the Objects View. The graphs for the SS14 Group display the dramatic drop in average item score for the OBJ Objects Scale and the corresponding rise in the preference for the Numbers View, as measured by the NBR Scale, over the period of the three weeks' intervention teaching which introduced these students to algebra. In the six months following this period there was some regression towards an Objects View.

The contrast between the groups on whether or not they favoured an Objects View of symbols or a Numbers View emphasized that the development of an understanding of algebraic symbols as representing numbers rather than objects appeared to be beneficial for ensuring substantial progress in the algebraic tasks assessed by the research instrument.

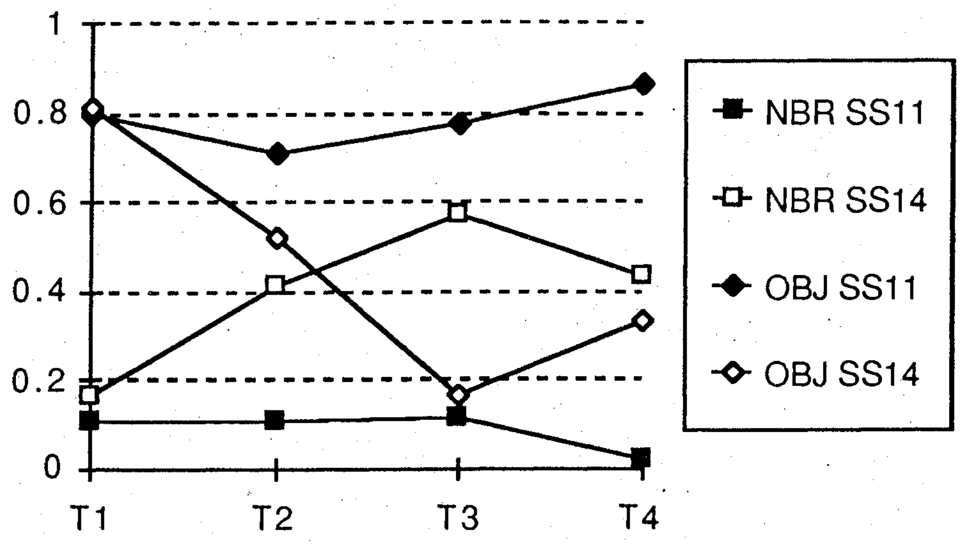

Figure 6: Average scores/item for SS11 and SS14 Groups on NBR and OBJ Scales

SUMMARY OF FINDINGS

An important problem for many teachers of students beginning algebra is the question of why, in the same class, some students do not show much or any progress while other students do. In Test 1, Group SS14 scored significantly better on test total than Group SS11, although the difference between the mean scores was only about 4 points. There were no other significant differences at this pre-algebra stage. By Test 2, significant differences on test total and eight cognitive measures had developed.

Group SS14 progressed more rapidly towards acquiring the concept that algebraic symbols stood for numbers which could vary than did their counterparts. Thus, empirical evidence indicated that it was student views of the meaning of the symbols which identified significant differences between those who were on the way to improvement and those who were not. The SS14 students also acquired more quickly the basic skills required to substitute in simple expressions and solve simple equations.

Group SS11 misunderstood more persistently such conventions as the conjoining process in algebra, and tended more to miss the point of problems set. This indicated that they were not coping with the new ideas that were being placed before them. Significant differences between the groups were registered in each following test on more than 20 cognitive measures.

The failure of Group SS11 students to succeed in interpreting algebraic expressions (e.g., '$3y + 5$') sufficiently well to enable them to make substitutions correctly was accompanied by failure to succeed with test items requiring them to use the variable concept, as in comparing the values of two expressions (e.g., '$2n$' and '$n + 2$'). The overall outcomes for Group SS11 pointed to the likelihood that one of the reasons why they had not progressed in the development of an understanding of algebraic symbols as numerical variables was that they had not learnt to interpret algebraic expressions. A theoretical argument supporting this finding is that one needs to reduce the memory load required for simply understanding the meaning of a single algebraic expression before one can carry out cognitive tasks involving relationships between algebraic expressions.

Success in solving an equation with one operation on one variable (viz., '$3a = 36$') was a prelude to success in finding conditions for the equality of two algebraic expressions. It seemed, moreover, that the memory load needed to solve the easier case of equality had to be reduced before more complex equality problems could be solved. Group SS11 recorded a gradual improvement in solving the given equation, but this did not necessarily result in increased success with the other equality problems.

The contrast between the groups also indicated that the development of an understanding that algebraic symbols represent numbers rather than objects appeared to be beneficial to ensure progress in completing algebraic tasks successfully.